Control Systems : Set 8 : Loopshaping (4) - Solutions

Prob 1 | For the closed-loop transfer function

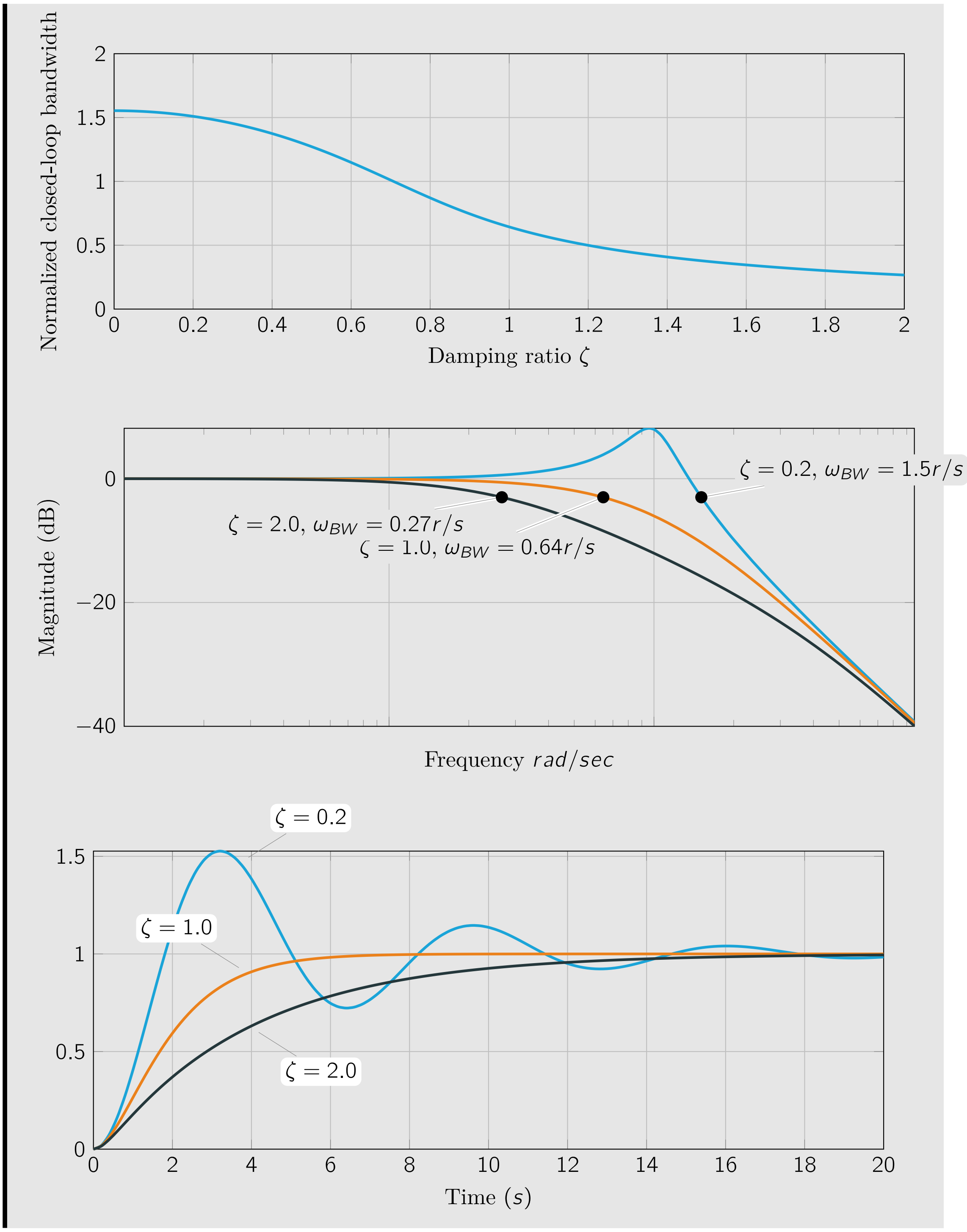
$$T(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

derive the following expression for the bandwidth ω_{BW} of $T(s)$ in terms of ω_n and ζ

$$\omega_{BW} = \omega_n\sqrt{1 - 2\zeta^2 + \sqrt{2 + 4\zeta^4 - 4\zeta^2}}$$

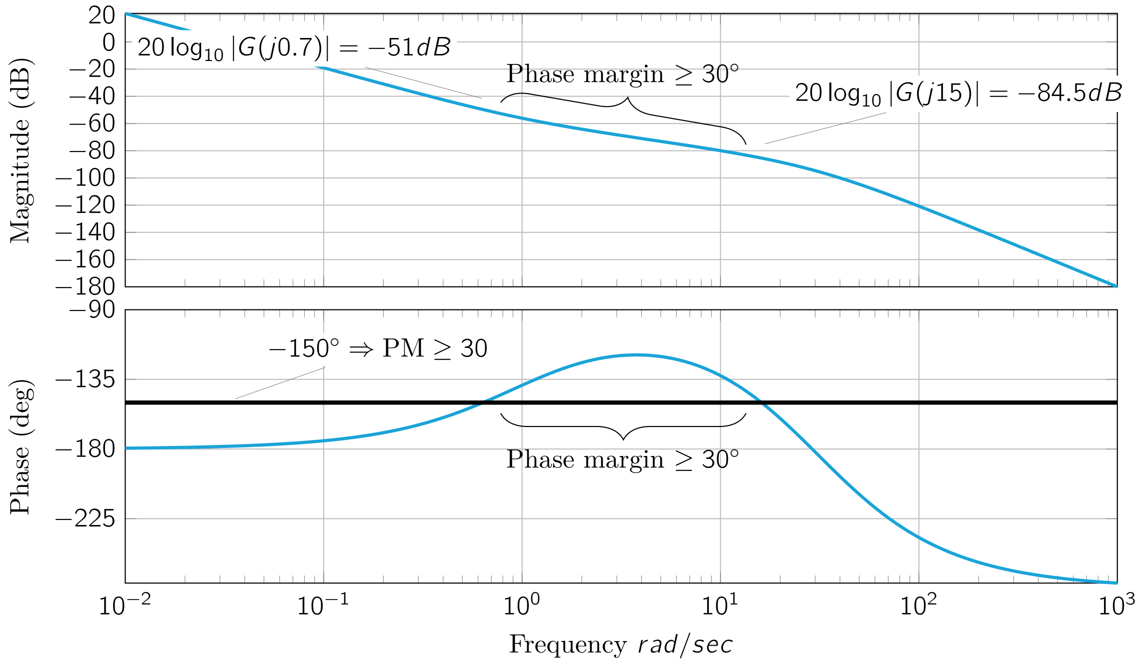
Assuming $\omega_n = 1$, use Matlab to plot ω_{BW} for $0 < \zeta < 1$.

> Bandwidth is defined as the frequency at which the magnitude drops to $\frac{1}{\sqrt{2}}$ of the DC value.
>
> $$|T(j\omega_{BW})| = \frac{1}{\sqrt{2}} = \frac{\omega_n^2}{(j\omega_{BW})^2 + 2\zeta\omega_n j\omega_{BW} + \omega_n^2}$$
>
> Define $x = \omega_{BW}/\omega_n$
>
> $$\begin{aligned} |T(jx)| = \frac{1}{\sqrt{2}} &= \left|\frac{1}{(jx)^2 + 2\zeta jx + 1}\right| \\ &= \frac{|1 - x^2 - \zeta x2j|}{x^4 + 4x^2\zeta^2 - 2x^2 + 1} \\ &= \frac{|1 - x^2 - \zeta x2j|}{x^4 + 4x^2\zeta^2 - 2x^2 + 1} \\ &= \frac{\sqrt{(1 - x^2)^2 + (\zeta x2)^2}}{x^4 + 4x^2\zeta^2 - 2x^2 + 1} \\ &= \frac{\sqrt{x^4 + 4x^2\zeta^2 - 2x^2 + 1}}{x^4 + 4x^2\zeta^2 - 2x^2 + 1} \\ &= \frac{1}{\sqrt{x^4 + 4x^2\zeta^2 - 2x^2 + 1}} \\ &\Leftrightarrow \\ \frac{1}{2} &= \frac{1}{x^4 + 4x^2\zeta^2 - 2x^2 + 1} \\ 2 &= x^4 + 4x^2\zeta^2 - 2x^2 + 1 \end{aligned}$$
>
> The quadratic equation then gives the desired solution.
> We plot the relationship between the damping ratio ζ and the normalized bandwidth of the system below. Notice that a lower damping ratio results in a higher bandwidth - less robust and more oscillation, but a faster response. To emphasize this point, we show below three plots for $\zeta = 0.2$, 1.0 and 2.0.


Normalized closed-loop bandwidth
Damping ratio ζ
Magnitude (dB)
ζ = 0.2, ω_BW = 1.5r/s
ζ = 2.0, ω_BW = 0.27r/s
ζ = 1.0, ω_BW = 0.64r/s
Frequency rad/sec
ζ = 0.2
ζ = 1.0
ζ = 2.0
Time (s)

Prob 2 | The Bode plot of the following system for $K = 1$ is given below

$$KG(s) = \frac{K(s+1)}{s^2(s+30)^2}$$



a) Determine the range of gains K that will yield a phase margin of at least 30°.

b) What is the maximum possible closed-loop bandwidth that satisfies $PM \geq 30°$?

c) Use Matlab to confirm your finding.

A phase margin of 30° requires that the phase is larger than $-150°$. We see from the Bode plot that this happens in the frequency range of $0.7 < \omega < 15$.
The corresponding gains at these frequencies are $G(j0.7) = -50$dB and $G(j15) = -84$dB, corresponding to and allowed range of gains

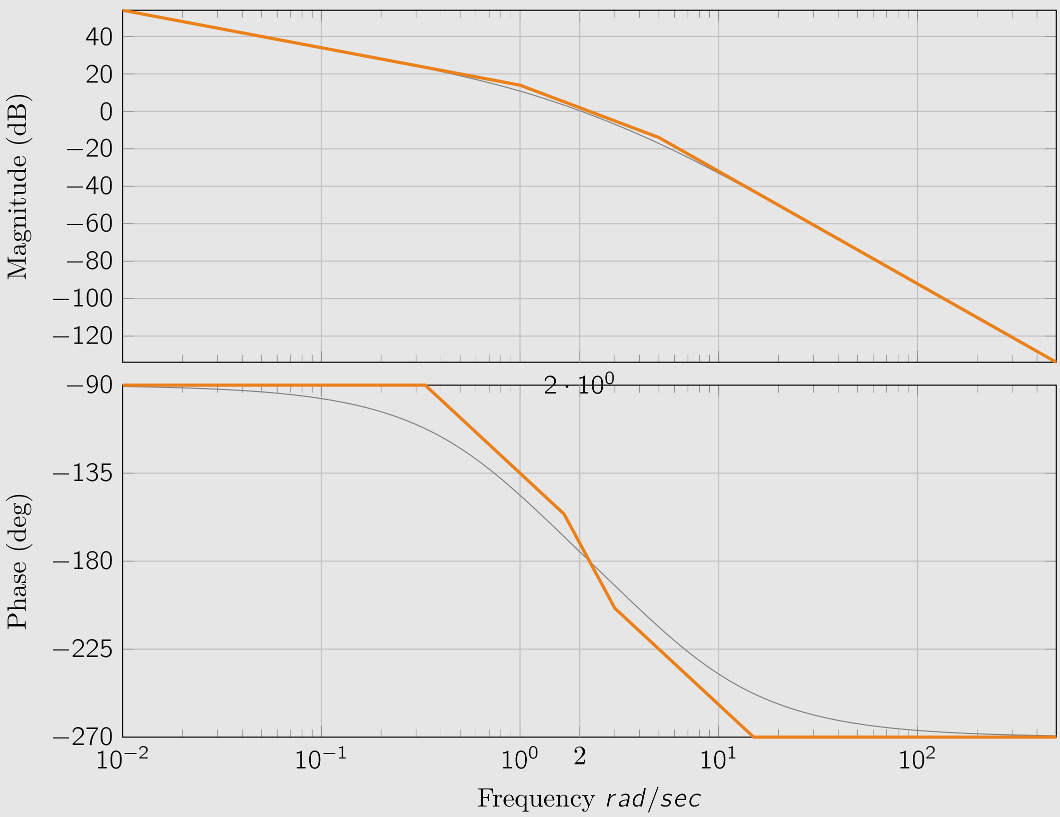
$$50dB \approx 300 < K < 16,000 \approx 84dB$$

In this case, the bandwidth is larger for larger values of K, so the maximum crossover frequency is $\omega = 15$ for a gain of $K = 16,000$. The resulting bandwidth will be within a factor of two of this, and if we compute it in Matlab, we will see that it is 25rad/sec.

Prob 3 | Design a lead compensator $D_c(s)$ with unity DC gain so that $PM \geq 40°$ using Bode plot sketches, then verify your design using Matlab. What is the approximate bandwidth of the system?

$$G(s) = \frac{5}{s(s+1)(s/5+1)}$$

We first sketch a straight-line bode plot approximation.



We notice that the phase is approximately $-180°$ when the gain is $0dB$, which occurs around $2rad/sec$ and which implies that our phase margin is approximately $0°$. This tells us that we need to add $40°$ of phase at $2rad/sec$.

Choose α to give an additional $60°$ to be safe, and $\omega_{\max}$ at $2r/s$

$$\alpha = \frac{1-\sin 60°}{1+\sin 60°} = 0.07$$

$$\omega_{\max} = 2 = \frac{1}{T_D\sqrt{\alpha}} \qquad \rightarrow \qquad T_D = 1.9$$

The resulting controller is then

$$D_c(s) = \frac{1.9s + 1}{0.13s + 1}$$

This should give us a phase margin of around $40°$ and a bandwidth a little larger than $2r/s$. However, if we check this with Matlab, we see that we get a PM of $13.8°$ and a bandwidth of $8r/s$. This is happening because the pole at $5r/s$ is causing our phase to drop at our new crossover frequency. The solution to this is to move ω_{max} a little to the right, say around $\omega_{\mathrm{max}} = 5 - 10r/s$.
We try again with $\omega_{\mathrm{max}} = 5$, which gives the control law

$$D_c(s) = \frac{0.76s + 1}{0.053s + 1}$$

which results in a phase margin of $42.4°$ and a bandwidth of $5.6r/s$.

Prob 4 | The inverted pendulum has a transfer function given by

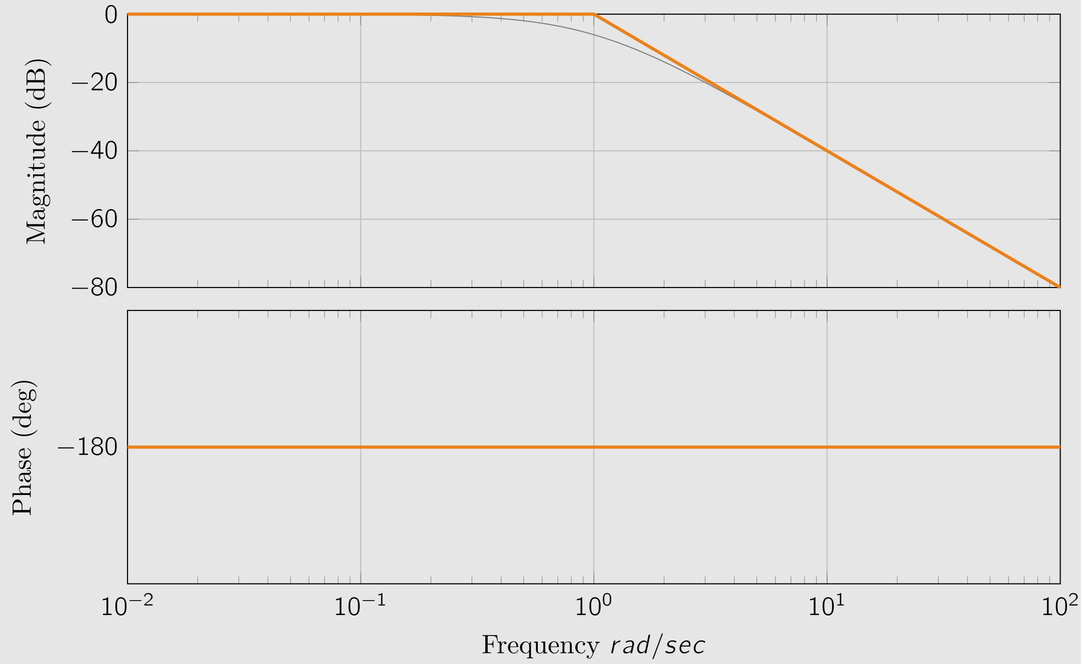
$$G(s) = \frac{1}{s^2 - 1}$$

a) Design a lead compensator to achieve a PM of 30° and a bandwidth around $1 r/s$ using a Bode plot sketch, then verify and refine your design using Matlab.

b) Could you obtain the frequency response of this system experimentally?

> The open-loop system has a RHP pole at 1, and so doing experiments on the open-loop system will result in unstable behaviour.

We first sketch the bode plot



We see that the phase everywhere is $-180°$. Therefore, to get a phase margin of 30°, we need a phase increase of 30°

$$\alpha = \frac{1 - \sin 30°}{1 + \sin 30°} = \frac{1}{3}$$

We just place the crossover at $\omega_{\text{max}} = 1$

$$T_D = \frac{1}{\sqrt{\alpha}} = \sqrt{3}$$

This gives the control law

$$D_c(s) = K\frac{\sqrt{3}s + 1}{s/\sqrt{3} + 1}$$

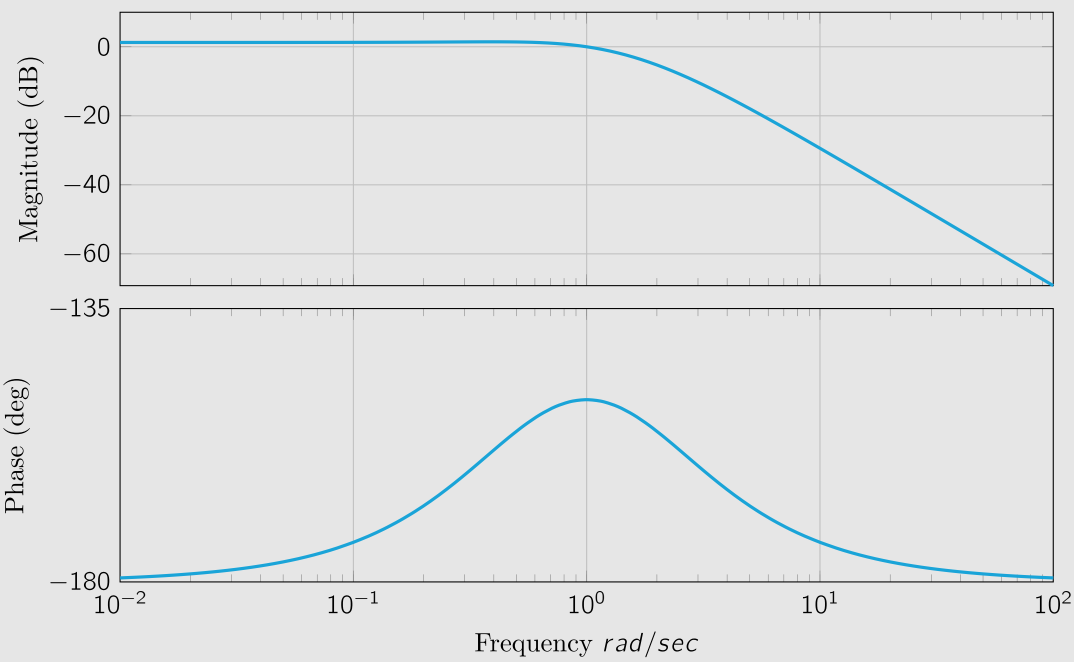
choose the gain K so that the crossover frequency is $1r/s$

$$|D_c(1j)G(1j)| = \left|K\frac{1}{(1j)^2 - 1} \cdot \frac{\sqrt{3}j + 1}{j/\sqrt{3} + 1}\right| = 1$$
$$\Rightarrow K = \frac{2}{\sqrt{3}}$$

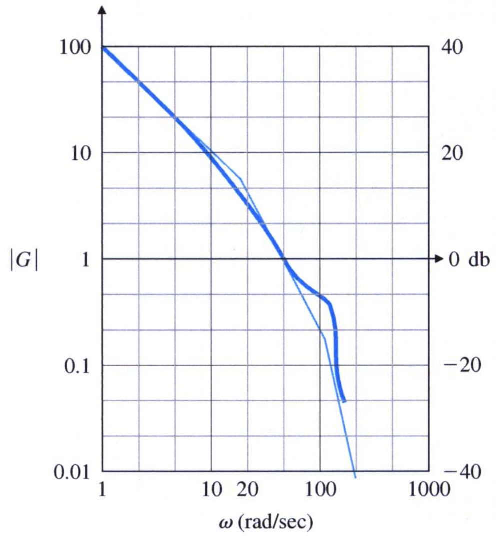
The control law is then

$$D_c(s) = \frac{2}{\sqrt{3}} \cdot \frac{\sqrt{3}s + 1}{s/\sqrt{3} + 1}$$

The bode plot of $D_c(s)G(s)$ is shown below, which has a phase margin of 30° and a crossover frequency of $1r/s$.



Prob 5 | The frequency response of a plant in a unity-feedback configuration is sketched in the figure below. Assume that the plant is open-loop stable and minimum-phase.



a) What is the velocity constant K_v for the system as drawn?

The slope at low frequency is −20dB/dec, and therefore the system is Type 1. At low frequencies, the transfer function will approximately be

$$G(j\omega) \approx \frac{K_v}{j\omega}$$

From the plot at $\omega = 1$, we see that $K_v = 100$

b) What is the damping ratio of the complex poles at $\omega = 100$?

The damping ratio is responsible for the size of the resonance peak at $\omega = 100$, which we estimate from the figure to be about $-20/3 \approx 6.5$dB.

Consider the magnitude of a second order system evaluated at the natural frequency:

$$G(s) = \frac{1}{\left(\frac{s}{\omega_n}\right)^2 + 2\zeta\left(\frac{s}{\omega_n}\right) + 1}$$

$$|G(j\omega_n)| = \frac{1}{2\zeta}$$

From which we can estimate the damping ratio

$$\frac{1}{2\zeta} = 10^{6.5/20} \qquad \Leftrightarrow \qquad \zeta = 0.24$$

c) Estimate the PM of the system as drawn. (Hint: Bode phase-gain relationship)

Since the plant is a minimum phase system, we can apply Bode's approximate gain-phase relationship.

At crossover, we have a slope of -40dB/dec, which implies a phase of $-180°$ and a phase margin very close to zero.